The hourglass problem

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(Received 30 April 1984; accepted for publication 1 August 1984)

I. INTRODUCTION

In the well-known beginning physics textbook, *Fundamentals of Physics, 2nd Ed., Extended* by Halliday and Resnick, the following question[1] is posed for consideration by the student:

> An hourglass is being weighed on a sensitive balance, first when sand is dropping in a steady stream from the upper to the lower part and then again after the upper part is empty. Are the two weights the same or not? Explain your answer.

We find this question rather interesting because the answer which springs to mind most readily is wrong! It is tempting to state, since the center of mass of the hourglass system has clearly moved downward, its acceleration is downward so that the net force on the system is also downward and thus the reading of the scale will be *less* than when the sand is at rest. It is quite a surprise to find that this is incorrect—the force exerted by the scale is actually *larger* than when the sand is at rest. The center of mass is actually accelerating *upwards* during most of the process.

We can understand this result best by actually calculating the center of mass acceleration of the hourglass. In Fig. 1 we idealize the hourglass. Let y_2 be the height of the sand in the top portion and y_1 the height in the bottom. The height of the constriction is a. If $A(y)$ is the cross-sectional area of the hourglass at the height y and ρ the density of the sand, then the center of mass position is given by

$$MY_{\text{c.m.}} = \int_0^{y_1} y\rho A(y)dy + \int_a^{y_2} y\rho A(y)dy + C, \tag{1}$$

where M is the total mass of the hourglass and C is some constant which takes into account the sand which is in the process of falling from top to bottom and the construction of the hourglass excluding the sand. By differentiating Eq. (1) with respect to time we obtain

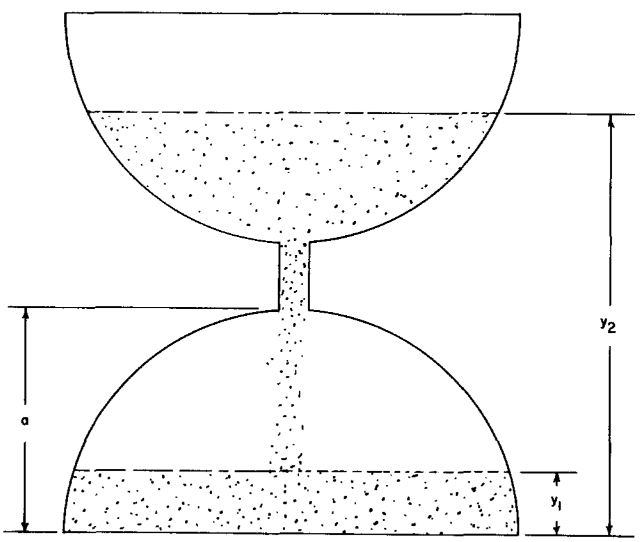

Fig. 1. Hourglass in quasi-steady-state motion. The figure defines the quantities a, y_1, and y_2.

$$MV_{\text{c.m.}} = \rho\left[A(y_1)y_1\left(\frac{dy_1}{dt}\right) + A(y_2)y_2\left(\frac{dy_2}{dt}\right)\right]. \tag{2}$$

We shall introduce the rate of mass flow, K:

$$K = \frac{dm}{dt} = \rho A(y_1)\frac{dy_1}{dt} = -\rho A(y_2)\frac{dy_2}{dt}. \tag{3}$$

Then Eq. (3) can be rewritten.

$$MV_{\text{c.m.}} = K(y_1 - y_2). \tag{4}$$

(Note that since $y_1 < y_2$, $V_{\text{c.m.}}$ is negative as it must be.) We now differentiate once more to obtain the acceleration of the center of mass. Using Newton's second law, we find

$$F_{\text{scale}} - Mg = MA_{\text{c.m.}} = K\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + (y_1 - y_2)\left(\frac{dK}{dt}\right). \tag{5}$$

Let ΔF be the change in the scale reading $(F_{\text{scale}} - Mg)$ and, using Eq. (3), we find

$$\Delta F = \left(\frac{K^2}{\rho}\right)\left(\frac{1}{A(y_1)} + \frac{1}{A(y_2)}\right) - (y_2 - y_1)\left(\frac{dK}{dt}\right). \tag{6}$$

In our experiment the flow rate is fairly constant, so the second term in Eq. (6) is negligible. The final result is then

$$\Delta F = (K^2/\rho)[1/A(y_1) + 1/A(y_2)]. \tag{7}$$

From Eq. (7) one notes that ΔF is always positive—the center of mass is accelerating upwards although it is moving downwards! During the transient at the beginning of the motion, just after the sand has begun to fall but has not yet struck the bottom, the center of mass has a large acceleration downwards and thus acquires a downward velocity. Once the quasi-steady-state condition has been achieved, however, the acceleration is reversed and from that instant on is upward.

An alternate explanation often given is that the loss in weight of the hourglass due to the falling sand is exactly balanced by the impact force when the sand strikes the bottom. Thus the hourglass weighs the same whether the sand is falling or not. This analysis errs by neglecting the nonzero speed of the sand as it leaves the top part and enters into free fall as well as neglecting some forces exerted by the sand on the hourglass in the upper portion as the sand accelerates towards the neck. The initial speed decreases the total time of fall and results in less total mass being in free fall at any instant. (As before, constant mass flow is assumed.) The net result is that the impact force is greater than the amount of weight in free fall by Kv_0, where v_0 is the speed of the sand at the beginning of free fall and K is the rate of mass flow given by Eq. (3). In addition to this, however, we must also consider the acceleration of the sand as it moves toward the opening. The net result of this is a momentum change of $K(v_0 - v)$ where $v = -dy_2/dt$ is the speed of the top of the sand. When the force causing this is

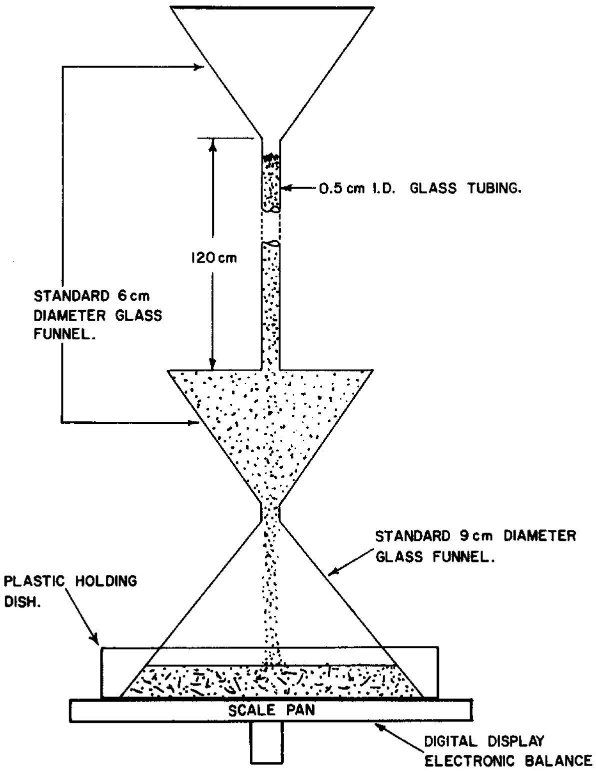

Fig. 2. Sketch of the hourglass device designed to be used as a demonstration of this effect.

combined with the above effect, we obtain Kv which is equivalent to Eq. (7) if the first term is neglected and use is made of Eq. (3).

Another approach can be taken which does not consider the process so much in detail. Let us just note that the net result of the movement of the sand in time dt is to transfer sand of mass dm from the top of the hourglass where it has speed v to the bottom where it is at rest. (We approximate $dy_1/dt = 0$.) The force required to effect that change is $\Delta F = v\, dm/dt = Kv$ as obtained before.

II. EXPERIMENT

We thought this result to be so anti-intuitive that it would make an interesting demonstration device. Also, this is a good simple example of the application of Newton's second law to a nonrigid system having a rather complicated internal motion. After considerable experimentation, we finally settled on the device diagramed in Fig. 2.

The area $A(y_2)$ corresponds to the 0.5-cm-i.d. glass tube and was chosen to be very small so that ΔF would be a measurable number. (For an ordinary hourglass, the effect would be unnoticeable.) The height of 120 cm gives about 10–15 s of flow time—enough to make reliable readings. To estimate ΔF, we assumed $A(y_2) = \pi d^2/4$, where $d = 0.5$ cm, $A(y_1) =$ very large, $K = 3.7$ g/s, and $\rho = 1.5$ g/cm^3. This, when used with Eq. (7), yields $\Delta F = 48$ dyn corresponding to a balance reading of 49 mg.

Thus we see that the effect is very small even for our apparatus designed to produce a large effect. In order to observe it one must have a sensitive balance with no overhead obstructions, capable of giving milligram accuracy with a load of the order of hundreds of grams. This is a difficult requirement, but the Mettler P C 440 balance[2] does the job very nicely. For demonstration purposes, the static weight can be tared out and the balance will then display ΔF directly in a brightly lit digital display.

The "sand" is also critical for good results. It must flow freely and should have a large density to make ΔF large. Ordinary sand does not work well, perhaps because of the jagged edges on the individual grains. We found that the tiny glass beads[3] used by opticians to heat up eyeglass frames prior to adjustment worked satisfactorily.

Perhaps the most difficult part of the development was learning how to start the flow without disturbing the delicate balance of the system. We settled on a small piece of dental wax[4] which gently plugged the bottom opening. Upon heating, with a cigarette lighter or Bunsen burner, the wax melts slightly and drops out allowing the flow to begin.

The results obtained from the aforementioned apparatus are qualitatively quite convincing. After the initial start, the display of the balance was always positive as required by the simple theory presented above. The display was rather erratic, however, possibly because of the nonsteady flow of the sand. Because of the erratic readings, it is not possible to give a quantitative comparison with theory, but the values obtained (from 30 to 80 mg) were certainly consistent with our model. If the necessary very accurate balance is available, this demonstration can provide good experimental verification of a rather generally misunderstood physical system.

ACKNOWLEDGMENTS

We would like to thank the Department of Microbiology for allowing us to use their balance, and B. A. Scott and Hubert Lloyd for their help. We also greatly appreciate the skillful help of our glassblower, Robert Clark, who made the apparatus, and the insights of our colleague, R. Dean Ayers.

[1]D. Halliday and R. Resnick, *Fundamentals of Physics, 2nd Ed., Extended* (Wiley, New York, 1981), p. 164, question 10. A similar problem is posed by R. M. Eisberg and L. S. Lerner, *PHYSICS Foundations and Applications* (McGraw-Hill, New York, 1981), Vol. I, p. 159, problem 4-48.

[2]The Mettler P C 440 balance is capable of weighing loads up to 400 g to milligram accuracy and is available from the Fisher Scientific Company.

[3]These beads are readily obtainable from any optical supplier.

[4]Most pharmacies carry this wax which is used to lubricate dental braces.